

# Bound on the slope of steady water waves with favorable vorticity

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## Abstract

We consider the angle  $\theta$  of inclination (with respect to the horizontal) of the profile of a steady 2D inviscid symmetric periodic or solitary water wave subject to gravity. Although  $\theta$  may surpass  $30^\circ$  for some irrotational waves close to the extreme wave, Amick [Ami87] proved that for any irrotational wave the angle must be less than  $31.15^\circ$ . Is the situation similar for periodic or solitary waves that are not irrotational? The extreme Gerstner wave has infinite depth, adverse vorticity and vertical cusps ( $\theta = 90^\circ$ ). Moreover, numerical calculations show that even waves of finite depth can overturn if the vorticity is adverse. In this paper, on the other hand, we prove an upper bound of  $45^\circ$  on  $\theta$  for a large class of waves with favorable vorticity and finite depth. In particular, the vorticity can be any constant with the favorable sign. We also prove a series of general inequalities on the pressure within the fluid, including the fact that any overturning wave must have a pressure sink.

## 1 Introduction

The celebrated “extreme Stokes wave” has angle (with respect to the horizontal) exactly  $30^\circ$  at its crest, as originally conjectured by Stokes himself [Sto47]. This is the limiting wave, singular at its crest, of the curve  $\mathcal{K}$  of irrotational waves that bifurcates from a trivial (flat laminar) wave. However, it was a surprise when numerical calculations [CS80, SM73] indicated that the angle  $\theta$  may surpass  $30^\circ$  for some waves on  $\mathcal{K}$  that are very close to the extreme wave. This fact was subsequently proven by McLeod in [McL97]. The maximum angle does not occur at the crest but very close to it. In a remarkable paper [Ami87] Amick proved that for any irrotational wave the angle must be less than  $31.15^\circ$ .

For waves that are not irrotational, there are no known analytical bounds on  $\theta$ , even  $90^\circ$ . Indeed, with adverse vorticity, the crests of Gerstner’s explicit waves in deep water can have cusps with a  $90^\circ$  angle, the extreme case being a cycloid, and numerical calculations in finite depth show that waves can be quite steep and even overturn [TdSP88, KS08, VO78]. In all of these cases, the vorticity is *adverse* (positive in our formulation). On the other hand, the main purpose of the present paper is to prove an upper bound of  $45^\circ$  for a large class of waves with *favorable* vorticity (negative in our formulation) in finite depth. The waves to which our bound applies include at least a large part of the well-known bifurcating curve.

We denote the velocity by  $(U, V)$ , the vorticity by  $\omega = V_x - U_y$ , and the (relative) stream function by  $\psi$ , where  $\psi_y = U - c$ ,  $\psi_x = V$ . The vorticity  $\omega$  depends only on the stream function,  $\omega(x - ct, y) = \gamma(\psi(x - ct, y))$ . By a streamline we mean a level curve of  $\psi$ , i.e. a particle path *in a frame moving with the wave*. Our main result, somewhat informally stated, is as follows.

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**Theorem 1.** *Let  $\mathcal{C}$  be a connected set (containing a trivial wave) of symmetric periodic finite-depth water waves traveling at speed  $c$  with a single crest and trough per period for which  $U < c$  (non-stagnation) and for which the streamlines (in the moving frame) are strictly decreasing from crest to trough. We assume that  $\gamma \leq 0$ ,  $\gamma' \leq 0$ ,  $\gamma'' \leq 0$ . At least until  $(U - c)\gamma = g$  at the troughs, the waves in  $\mathcal{C}$  that bifurcate from a trivial wave have angle strictly less than  $45^\circ$ . (The trivial wave satisfies  $(U - c)\gamma < g$  everywhere.)*

*The same statement is true for symmetric solitary waves (instead of periodic waves) at least until  $(U_\infty - c)\gamma = g$  on the surface, where  $U_\infty(y) = \lim_{x \rightarrow \pm\infty} U(x, y)$ .*

In fact, we will prove a bound that is strictly less than  $45^\circ$ , although the bound depends on the wave; see Corollaries 7 and 13.

In [Ami87], Amick first shows that  $\theta < \pi/2\beta \approx 38.2^\circ$ , where  $\beta = (9 + \sqrt{97})/8$ , and then uses a different argument to improve this bound to  $31.15^\circ$ . His first step is somewhat related to our arguments for waves with vorticity, and we outline it now. Working with the Nekrasov formulation, which is valid only in the irrotational case, Amick essentially considers

$$f_\alpha = \operatorname{Re}[(c - U) + iV]^\alpha = [(U - c)^2 + V^2]^{\alpha/2} \cos(\alpha\theta),$$

which is the real part of an analytic function of the complex variable  $z = x + iy$ , where  $\alpha \geq 1$  is a parameter. The bound  $|\theta| < \pi/2\alpha$  will follow if we can show that  $f_\alpha > 0$ . Since  $f_\alpha = (c - U)^\alpha > 0$  at crests and troughs, it is enough for  $f_\alpha$  to be decreasing from crest to trough along the free surface, i.e. for its tangential derivative

$$W_\alpha = \frac{(U - c)\partial_x + V\partial_y}{(U - c)^2 + V^2} f_\alpha,$$

which is again the real part of an analytic function of  $z$ , to have an appropriate sign on each half period. By combining maximum principle arguments with a continuation in the parameter  $\alpha$ , Amick deduces that  $W_\alpha$  has the appropriate sign (and hence  $|\theta| < \pi/2\alpha$ ) for  $1 \leq \alpha \leq \beta$ . Toland and Plotnikov [PT02] subsequently gave a different proof, also depending on the Nekrasov formulation and complex variable techniques, that the angle is less than  $45^\circ$ .

There are considerable difficulties in extending Amick's arguments to waves with vorticity. First, with a general vorticity, the water wave problem cannot be reformulated as a Nekrasov-type integral equation on the boundary. More significantly, complex function theory no longer guarantees that the functions  $f_\alpha$  and  $W_\alpha$  are harmonic. Indeed, for a general  $\alpha$ , formulas for their Laplacians contain many terms of seemingly indeterminate sign. Thus Amick's application of the maximum principle to  $W_\alpha$  seems out of reach. Nevertheless, we are able to distill some of Amick's ideas in the special case  $\alpha = 2$ , where the formulas are simpler, and under the additional assumption  $\gamma, \gamma', \gamma'' \leq 0$  on the vorticity function. Instead of varying  $\alpha$ , we use a continuation argument along a connected set  $\mathcal{C}$  of solutions, an option which Amick also explores in [Ami87].

Section 2 is devoted to the main theorem (Theorem 3), which is a more precise version of Theorem 1 that specifies the notion of connectedness. The key to the proof is that  $U_x$  has a strict sign between each crest and trough. In order to prove this fact, we look at the possibility that, for some wave along  $\mathcal{C}$ ,  $U_x$  might vanish somewhere away from the crest and trough. Then we make use of the Hopf maximum principle applied to  $U_x/(U - c)$  and the slope  $V/(U - c)$  together with the boundary conditions to prove certain inequalities. The quantity  $g - (U - c)\gamma$  plays an important role. We note that the product  $(U - c)\gamma$  is the vertical component of what is sometimes called the "vortex force" [Saf92].

Construction of the connected bifurcation set goes back to [KN78] in the irrotational case and [CS04] in the case of general vorticity. In Section 3 we discuss the relationship between our basic result and the bifurcation set of solutions that we know exist.

In Section 4, Theorem 1 is proven for the case of solitary waves.

Section 5 is devoted to a series of inequalities on the pressure inside the fluid in the absence of stagnation points. Some of them have been previously known and others are new facts but in any case we prove them in enough generality to permit overturning waves. All of them are based on the maximum principle applied to various quantities. One of them is used in the proof of the main theorem. Allowing for stagnation points inside the fluid but not on the free surface, we also prove that every overturning wave must have a pressure sink.

We thank John Toland for informing us of the reference [PT02].

## 2 Bound on the slope

For the rest of this paper it is convenient to denote the relative velocity by  $u = U - c$  and  $v = V$ . Except in Sections 4 and 5.2, we consider symmetric  $2L$ -periodic water waves with vorticity. In a frame moving with the wave they are described as solutions of

$$uu_x + vu_y = -P_x \quad \text{in } -d < y < \eta(x), \quad (1a)$$

$$uv_x + vv_y = -P_y - g \quad \text{in } -d < y < \eta(x), \quad (1b)$$

$$u_x + v_y = 0 \quad \text{in } -d < y < \eta(x), \quad (1c)$$

$$P = P_{\text{atm}} \quad \text{on } y = \eta(x), \quad (1d)$$

$$v = \eta_x u \quad \text{on } y = \eta(x), \quad (1e)$$

$$v = 0 \quad \text{on } y = -d, \quad (1f)$$

with  $u, \eta$  even in  $x$ ,  $v$  odd in  $x$ , and where  $u, v, \eta, P$  are  $2L$ -periodic in the  $x$  variable. We take  $\eta(x)$  to have mean value zero, so that  $d$  is the average depth. The “atmospheric” pressure  $P_{\text{atm}}$  and depth  $d > 0$  are constants. As for regularity, we will assume  $\eta \in C_{\text{per}}^4[-L, L]$  and  $u, v, P \in C_{\text{per}}^3(\overline{\Omega})$ , where

$$\Omega = \{(x, y) \in \mathbb{R}^2 : -L < x < L, -d < y < \eta(x)\}$$

denotes a period the fluid domain and “per” denotes  $2L$ -periodicity in  $x$ . In addition, we will always assume that

$$\sup_{\Omega} u < 0, \quad (2)$$

which in particular rules out stagnation points in the fluid where  $u = v = 0$ .

We say a wave is *trivial* if  $u, v, \eta, P$  depend only on the vertical variable  $y$ . This forces  $v \equiv 0$ ,  $\eta \equiv 0$ , and  $P = -gy$ , but does not place any new restrictions on the horizontal velocity  $u = u_0(y)$ , which at this point can be an arbitrary (negative) function of  $-d \leq y \leq 0$ .

Let  $\Omega^+$ ,  $S^+$ , and  $B^+$  denote half-periods of the fluid domain, free surface, and bed,

$$\Omega^+ = \{(x, y) \in \mathbb{R}^2 : 0 < x < L, -d < y < \eta(x)\},$$

$$S^+ = \{(x, \eta(x)) : 0 < x < L\},$$

$$B^+ = \{(x, -d) : 0 < x < L\}.$$

See Figure 1a. We assume that all nontrivial waves are strictly monotone in that

$$v > 0 \text{ in } \Omega^+ \cup S^+. \quad (3)$$

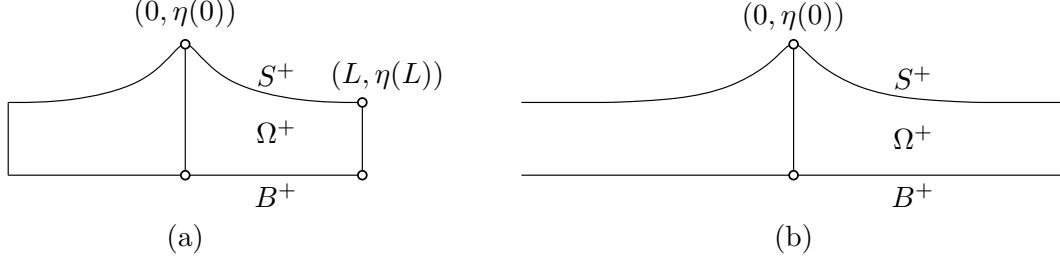


Figure 1: (a) The half-period  $\Omega^+$  of the fluid domain, together with the portions  $S^+$  and  $B^+$  of its boundary on the free surface and bed. (b) Analogous definitions for a solitary wave.

Using (2) and (3) in (1e), we see that the slope

$$\eta_x = \frac{v}{u} < 0 \quad \text{on } S^+.$$

That is, the free surface  $\eta$  is strictly decreasing between the crest at  $(0, \eta(0))$  and trough at  $(L, \eta(L))$ . All of the above assumptions are satisfied by the waves constructed in [CS04].

Thanks to incompressibility (1c), there exists a stream function  $\psi \in C^4_{\text{per}}(\overline{\Omega})$ , unique up to an additive constant, satisfying

$$\psi_y = u, \quad \psi_x = -v$$

in the fluid domain  $\Omega$ . By the kinematic boundary conditions (1e) and (1f),  $\psi$  is constant on both the free surface  $y = \eta(x)$  and the bed  $y = -d$ . We normalize  $\psi$  so that it vanishes on the free surface, and let  $m$  denote its value on  $y = -d$ , which is positive since by (2) we have  $\psi_y = u < 0$ . The assumption (2) also guarantees that the vorticity  $\omega = v_x - u_y$  can be expressed as

$$\omega = -\Delta\psi = \gamma(\psi)$$

for some function  $\gamma \in C^2[0, m]$  called the *vorticity function*; see for instance [CS04].

The main additional assumption that we will make throughout Section 2 is that the vorticity function  $\gamma$  satisfies the sign conditions

$$\gamma(\psi) \leq 0, \quad \gamma'(\psi) \leq 0, \quad \gamma''(\psi) \leq 0 \quad \text{for } 0 < \psi < m. \quad (4)$$

Note that (4) is satisfied whenever  $\gamma \leq 0$  is constant. For the case of a trivial wave with  $u = u_0(y) < 0$ , a short calculation shows that (4) is equivalent to

$$u_{0y} \geq 0, \quad u_{0yy} \leq 0, \quad u_0 u_{0yyy} - u_{0y} u_{0yy} \leq 0. \quad (5)$$

**Definition 2** (The water wave set  $\mathscr{W}$ ). Fixing the half-period  $L$ , we denote by  $\mathscr{W}$  the set of waves satisfying the above assumptions, or more precisely the set of 5-tuples  $(u, v, P, \eta, d)$  with  $d > 0$ ,  $\eta \in C^4_{\text{per}}[-L, L]$  and  $u, v, P \in C^3_{\text{per}}(\overline{\Omega})$  for which  $u, P, \eta$  are even in  $x$  and  $v$  is odd in  $x$ , which satisfy (1), (2), and (4), and which either satisfy the monotonicity condition (3) or are trivial.

We are interested in subsets  $\mathscr{C} \subset \mathscr{W}$  which are *connected* in some sense. In [CS04], this connectedness is expressed by first making a change of variables which transforms the problem into a periodic strip. For our purposes, a weaker notion of connectedness merely involving traces on the free surface will suffice. Consider the mapping

$$\tau: \mathscr{W} \rightarrow (C^0[0, L])^7, \quad \tau(u, v, P, \eta, d)(x) = (u, v, u_x, v_x, u_y, u_{xx}, u_{xy})(x, \eta(x)). \quad (6)$$

Our main result below is a more precise version of Theorem 1. It concerns subsets  $\mathcal{C}$  of  $\mathcal{W}$  which contain a trivial wave and for which  $\tau(\mathcal{C})$  a connected subset of  $(C^0[0, L])^7$ . One can check that the sets  $\mathcal{C}$  constructed in [CS04] satisfy this connectedness property.

**Theorem 3.** *Let  $\mathcal{C} \subset \mathcal{W}$  be a set of periodic water waves such that  $\tau(\mathcal{C})$  is a connected subset of  $(C^0[0, L])^7$ . If  $\mathcal{C}$  contains at least one trivial wave, and if all the waves in  $\mathcal{C}$  satisfy*

$$g - \gamma u > 0 \quad \text{at the trough } (L, \eta(L)), \quad (7)$$

*then the slope  $|v/u|$  of any streamline of any wave in  $\mathcal{C}$  is  $< 1$ .*

The proof of Theorem 3 depends on a couple of lemmas. The first one provides a sufficient condition for the upper bound on the slope.

**Lemma 4.** *If a wave in  $\mathcal{W}$  satisfies  $u_x < 0$  in  $\Omega^+ \cup S^+$ , then  $\max_{\overline{\Omega}} |v/u| < 1$ .*

*Proof.* Letting  $f = \frac{1}{2}(u^2 - v^2)$ , we first note that the desired inequality is equivalent to  $f > 0$  on  $\overline{\Omega^+}$ . By symmetry and periodicity,  $v = 0$  on the vertical lines  $x = 0$  and  $x = L$  as well as the bed  $B^+$ . Since  $\max_{\overline{\Omega}} u < 0$  by (2), we therefore have  $f = u^2/2 > 0$  on  $\partial\Omega^+ \setminus S^+$ . Differentiating, we find

$$uf_x + vf_y = (u^2 + v^2)u_x - \gamma uv < 0 \quad \text{on } \Omega^+ \cup S^+, \quad (8)$$

where we have used the identities  $u_x + v_y = 0$  and  $v_x - u_y = \gamma$  to eliminate derivatives of  $v$  in favor of derivatives of  $u$ . The expression in (8) is strictly negative in  $\Omega^+ \cup S^+$ , since  $u_x < 0$  by assumption, and  $u < 0$ ,  $v \geq 0$ , and  $\gamma \leq 0$  by (2), (3), and (4).

So suppose for the sake of contradiction that  $\min_{\overline{\Omega^+}} f \leq 0$ . Then by the previous comments the minimum could not be achieved on  $\partial\Omega^+ \setminus S^+$  but only at some point  $(x^*, y^*)$  in  $\Omega^+ \cup S^+$ . Now if  $(x^*, y^*)$  lies in  $\Omega^+$ , then it would be a critical point of  $f$ ; that is,  $f_x = f_y = 0$  at  $(x^*, y^*)$ . But then the left hand side of (8) would vanish, which is a contradiction. Finally if  $(x^*, y^*)$  lies on the surface  $S^+$ , then we would have

$$0 > (u^2 + v^2)u_x - \gamma uv = uf_x + vf_y = u \frac{d}{dx} f(x, \eta(x)) = 0 \quad \text{at } (x^*, y^*),$$

which is again a contradiction. Thus the minimum of  $f$  is positive.  $\square$

Now let  $\mathcal{W}_0$  denote the set of waves in  $\mathcal{W}$  for which  $u_x \leq 0$  on  $S^+$ . The set  $\mathcal{W}_0$  certainly contains all the trivial waves in  $\mathcal{W}$  because they satisfy  $u_x \equiv 0$ . Looking back at the definition (6) of the mapping  $\tau$ , we also see that a wave  $(u, v, P, \eta, d)$  in  $\mathcal{W}$  lies in  $\mathcal{W}_0$  if and only if  $\tau(u, v, P, \eta, d)$  lies in  $\tau(\mathcal{W}_0)$ . Our main tool to prove Theorem 3 is the following lemma.

**Lemma 5** (Bounds along  $S^+$ ).

- (a) *For every nontrivial wave in  $\mathcal{W}_0$ ,  $u_x < 0$  on  $\Omega^+ \cup S^+$ .*
- (b) *For every nontrivial wave in  $\mathcal{W} \setminus \mathcal{W}_0$ ,  $\inf_{\Omega^+} u_x/u < 0$  is achieved at a point  $z_0 \in S^+$  where*

$$g(u^4 - 4u^2v^2 - v^4) - \gamma u^3(u^2 + 5v^2) < 0. \quad (9)$$

*Proof.* Note that there are no derivatives in the expression (9). We begin by writing the identities that are obtained by differentiating the dynamic boundary condition (1d) once and then twice along the free surface  $S$ . Since the pressure  $P$  is constant along the free surface  $S$ , we have

$$0 = u \frac{d}{dx} P(x, \eta(x)) = uP_x + vP_y = -u(uu_x + vu_y) + v(uv_x + vv_y + g) \quad \text{on } S,$$

where we have used the kinematic boundary condition (1e) to eliminate  $\eta_x$  in favor of  $v$  and then the Euler equations (1a)–(1b) to eliminate  $P_x$  and  $P_y$ . Using incompressibility  $u_x + v_y = 0$  to eliminate  $v_y$  and the definition  $v_x - u_y = \gamma$  of the vorticity function  $\gamma$  to eliminate  $v_x$ , we are left with

$$(v^2 - u^2)u_x - 2uvu_y - gv - \gamma uv = 0 \quad \text{on } S. \quad (10)$$

Taking another derivative along the free surface, we find

$$\begin{aligned} 0 &= (u\partial_x + v\partial_y) \left[ (v^2 - u^2)u_x - 2uvu_y - gv - \gamma uv \right] \\ &= -2(u^2 + v^2)(u_x^2 + u_y^2) + gv u_x - gu u_y + (3uv^2 - u^3)u_{xx} + (v^3 - 3u^2v)u_{xy} \\ &\quad - \gamma((3u^2 + v^2)u_y - 2uvu_x + gu) + 2\gamma' u^2 v^2 - \gamma^2 u^2 \end{aligned} \quad \text{on } S, \quad (11)$$

where now we have also differentiated the identities  $u_x + v_y = 0$  and  $v_x - u_y = \gamma$  to eliminate the second partials of  $v$ .

Consider any nontrivial wave in  $\mathcal{W}$ . We will apply maximum principle arguments to the two functions

$$w := \frac{u_x}{u}, \quad s := \frac{v}{u}.$$

The second one is the slope of the streamlines. Now two tedious but elementary computations show that both of these functions satisfy elliptic equations, namely,

$$\Delta w + 2 \frac{u_x}{u} w_x + 2 \frac{u_y}{u} w_y = \gamma'' v \leq 0, \quad (12)$$

$$\Delta s + 2v \frac{\gamma - us_x}{u^2 + v^2} s_x + 2u \frac{\gamma - vs_y}{u^2 + v^2} s_y = 0. \quad (13)$$

Therefore the maximum principle implies that

$$I := \inf_{\Omega^+} w = \inf_{\partial\Omega^+} w, \quad \sup_{\Omega^+} s = \sup_{\partial\Omega^+} s$$

are not attained in  $\Omega^+$ . Now  $v > 0$  and hence  $s < 0$  on  $\Omega^+$  by (3). By symmetry,  $s = v = 0$  on  $B^+$ . Thus the Hopf lemma (strong maximum principle) yields the inequality

$$0 > s_y = -\frac{u_x}{u} = -w \quad \text{on } B^+,$$

so that  $w > 0$  on the bottom  $B^+$ . On the lateral boundaries  $x = 0$  and  $x = L$ , we have  $w = 0$  by symmetry. Thus  $I \leq 0$  and  $w \geq 0$  on all of  $\partial\Omega^+ \setminus S^+$ .

Suppose now that the minimum  $I$  is achieved on the surface  $S^+$ , say at some point  $z_0 := (x_0, \eta(x_0)) \in S^+$ . Then the tangential derivative  $\partial_x(w(x, \eta(x)))$  must vanish at  $x = x_0$ . Thus we have

$$0 = uw_x + vw_y = -uw^2 - \frac{v}{u}u_y w + \frac{v}{u}u_{xy} + u_{xx} \quad \text{at } z_0. \quad (14)$$

By the Hopf lemma we also know that

$$w_y(z_0) < 0 \quad \text{at } z_0. \quad (15)$$

Solving (14) for  $u_{xx}$  and (10) for  $u_y$  and plugging these values into (11) at the point  $z_0$ , we obtain the equation

$$\begin{aligned} w_y &= \frac{u_{xy}}{u} - \frac{u_y}{u} w \\ &= -\frac{u(u^2 + v^2)^2}{4v^3} w^2 + \frac{g(u^4 - 4u^2v^2 - v^4) - \gamma u^3(u^2 + 5v^2)}{4u^2v^2(u^2 + v^2)} w - v \frac{g^2 + g\gamma u - 4\gamma' u^4}{4u^3(u^2 + v^2)} \\ &:= Aw^2 + Bw + C \end{aligned} \quad (16)$$

at  $z_0$ , where the coefficients  $A, B, C$  are functions of  $u, v$  and  $\gamma$  evaluated at  $z_0$ . Note that  $v > 0$  at  $z_0 \in S^+$  thanks to our monotonicity assumption (3). Combining (16) and (15), we have

$$Aw(z_0)^2 + Bw(z_0) + C < 0. \quad (17)$$

Since  $\gamma(0), \gamma'(0) \leq 0, u < 0$  and  $v > 0$ , both of the coefficients  $A$  and  $C$  are strictly positive. In particular,  $w(z_0) \neq 0$  and hence  $w(z_0) < 0$ .

Consider a wave in  $\mathcal{W}_0$ . Then  $u_x \leq 0$  and hence  $w \geq 0$  and  $I = 0$ . However, we have just shown that  $I = 0$  cannot be attained on  $S^+$ , since at such a point  $z_0 \in S^+$  we would have to have  $w(z_0) < 0$ . This completes the proof of (a).

On the other hand for a wave in  $\mathcal{W} \setminus \mathcal{W}_0$ , we have shown that  $I < 0$  is attained at some point  $z_0 \in S^+$  where  $w(z_0) < 0$  and where (17) holds. The first and last terms in (17) being strictly positive, the middle term  $Bw(z_0)$  must be strictly negative. Therefore  $B > 0$ . This is exactly the same as the inequality (9) in (b).  $\square$

*Proof of Theorem 3.* Thanks to Lemma 4 and Lemma 5(a), the desired bound  $|v/u| < 1$  holds for any wave in  $\mathcal{W}_0$ . Thus to prove the theorem it suffices to show that  $\mathcal{C} \subset \mathcal{W}_0$ , or equivalently  $\tau(\mathcal{C}) \subset \tau(\mathcal{W}_0)$ . Now  $\tau(\mathcal{C})$  is connected, so to prove  $\tau(\mathcal{C}) \subset \tau(\mathcal{W}_0)$  it is enough to show that  $\tau(\mathcal{C}) \cap \tau(\mathcal{W}_0)$  is nonempty, relatively open, and relatively closed in  $\tau(\mathcal{C})$ . It is easy to see from the nonstrict inequality in the definition of  $\mathcal{W}_0$  that  $\tau(\mathcal{C}) \cap \tau(\mathcal{W}_0)$  is relatively closed, and it is nonempty since the trivial wave in  $\mathcal{C}$  lies in  $\mathcal{W}_0$ . Thus it remains to show that  $\tau(\mathcal{C}) \cap \tau(\mathcal{W}_0)$  is relatively open in  $(C^0[0, L])^7$ .

Assume the contrary. That is, there exists a sequence  $\tau(u_n, v_n, P_n, \eta_n, d_n)$  in  $\tau(\mathcal{C}) \setminus \tau(\mathcal{W}_0)$  for which

$$\tau(u_n, v_n, P_n, \eta_n, d_n) \longrightarrow \tau(u, v, P, \eta, d) \quad \text{in } (C^0[0, L])^7 \quad (18)$$

for some  $\tau(u, v, P, \eta, d)$  in  $\tau(\mathcal{C}) \cap \tau(\mathcal{W}_0)$ . This means that  $u_n, v_n$ , and their first and second partials all converge uniformly as functions of  $x$  along the free surface. Since the definitions of  $\tau$  and  $\mathcal{W}_0$  imply  $\tau(\mathcal{C}) \setminus \tau(\mathcal{W}_0) = \tau(\mathcal{C} \setminus \mathcal{W}_0)$  and  $\tau(\mathcal{C}) \cap \tau(\mathcal{W}_0) = \tau(\mathcal{C} \cap \mathcal{W}_0)$ , we have  $(u_n, v_n, P_n, \eta_n, d_n) \in \mathcal{C} \setminus \mathcal{W}_0$  and  $(u, v, P, \eta, d) \in \mathcal{C} \cap \mathcal{W}_0$ . Thus  $(u_n)_x \not\leq 0$  on  $S_n^+$  but  $u_x \leq 0$  on  $S^+$ .

Applying Lemma 5(b) to  $(u_n, v_n, P_n, \eta_n, d_n)$  for any fixed  $n$ , we know that the function  $u_{nx}/u_n$  achieves a negative minimum over  $\Omega_n^+$  at some point  $z_n = (x_n, \eta_n(x_n)) \in S_n^+$ , with  $0 < x_n < L$ , at which we have

$$\left[ g(u_n^4 - 4u_n^2v_n^2 - v_n^4) - \gamma_n u_n^3(u_n^2 + 5v_n^2) \right](z_n) < 0. \quad (19)$$

By compactness we can assume that  $z_x$  converges to some  $x_0$  with  $0 \leq x_0 \leq L$ . Set  $z_0 = (x_0, \eta(x_0))$ . The uniform convergence (18) implies that

$$(u_n, v_n, u_{nx}, v_{nx}, u_{ny}, v_{ny}, u_{nxx}, v_{nxx})(z_n) \longrightarrow (u, v, u_x, v_x, u_y, v_y, u_{xx}, v_{xx})(z_0) \quad (20)$$

as  $n \rightarrow \infty$ . Letting  $\gamma_n$  be the vorticity function of  $(u_n, v_n, P_n, \eta_n, d_n)$ , we also have

$$\gamma_n(0) = (v_{nx} - u_{ny})(z_n) \rightarrow (v_x - u_y)(z_0) = \gamma(0).$$

In case  $(u, v, P, \eta, d)$  is a trivial wave, with  $\eta \equiv 0$ ,  $v \equiv 0$ ,  $u(x, y) = u_0(y)$ , then (20) would imply  $u_n(z_n) \rightarrow u_0(0) < 0$  and  $v_n(z_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Taking  $n \rightarrow \infty$  in (19) therefore yields  $gu_0^4(0) - \gamma u_0^5(0) \leq 0$ . Factoring and recalling that  $u_0(0) < 0$ , we deduce that  $g - \gamma u_0(0) \leq 0$ . Since  $\gamma$  and  $u = u_0$  are independent of  $x$ , this in particular implies  $g - \gamma u \leq 0$  at the trough  $(L, \eta(L))$ , which contradicts (7).

So  $(u, v, P, \eta, d)$  must be nontrivial. Because

$$0 < u_{nx}(z_n) \rightarrow u_x(z_0) \leq 0,$$

we must have  $u_x(z_0) = 0$ . But Lemma 5(a) asserts that  $u_x(x, \eta(x)) < 0$  for all  $0 < x < L$ . Thus the only remaining possibilities are that  $z_0$  is the crest  $(0, \eta(0))$  or the trough  $(L, \eta(L))$ . In either case,  $v_n(z_n) \rightarrow v(z_0) = 0$  and  $u_n(z_n) \rightarrow u(z_0) < 0$  as  $n \rightarrow \infty$ . Sending  $n \rightarrow \infty$  in (19), we get  $u^4(g - \gamma u) \leq 0$  at  $z_0$ , which yields  $g - \gamma u(z_0) \leq 0$ . Because of the assumption (7),  $z_0$  cannot be the trough. So  $z_0$  must be the crest.

By Theorem 15(e) below, we know that  $\partial P / \partial n < 0$  along the free surface. Since  $\eta_x < 0$  for  $0 < x < L$ , this in particular implies that  $P_x < 0$  on  $S^+$ . But then by the basic equations (1a) and (1e), we have

$$\frac{d}{dx} u^2(x, \eta(x)) = 2uu_x + 2vu_y = -2P_x > 0 \quad \text{for } 0 < x < L.$$

This means that  $u$  is strictly monotone along  $S^+$ . Thus  $0 > u(0, \eta(0)) > u(L, \eta(L))$ , so that

$$g - \gamma u(L, \eta(L)) \leq g - \gamma u(0, \eta(0)) = g - \gamma u(z_0) \leq 0,$$

contradicting (7). This completes the proof.  $\square$

**Corollary 6.** *Theorem 3 remains true if (7) is replaced by  $u_{xx} \neq 0$  at the trough for nontrivial waves.*

*Proof.* Consider  $(u_n, v_n, P_n, \eta_n, d_n)$ ,  $(u, v, P, \eta, d)$ , and  $z_n = (x_n, \eta_n(x_n))$  as in the proof of Theorem 3, and assume that  $(u, v, P, \eta, d)$  is nontrivial and satisfies  $u_{xx} \neq 0$  at the trough. Following the preceding argument we deduce that  $x_n \rightarrow x_0$ ,  $z_0$  is either the crest or the trough,  $u_x(z_0) = v(z_0) = 0$ , and  $g - \gamma u(z_0) \leq 0$ . Since the functions  $u_{nx}/u_n$  are minimized at  $z_n$ , we also have

$$0 = (u_n \partial_x + v_n \partial_y) \frac{u_{nx}}{u_n} = u_n^{-2} (u_n v_n u_{nxy} - v_n u_{nx} u_{ny} + u_n^2 u_{nxx} - u_n u_{nx}^2) \quad \text{at } z_n.$$

Taking limits by use of (20) then yields  $u_{xx}(z_0) = 0$ . Thus by assumption the point  $z_0$  could not be the trough and so could only be the crest. Now plugging  $u_{xx} = u_x = v = 0$  into (11) evaluated at the crest, we obtain the equality

$$0 = -2u^2 u_y^2 - guu_y - \gamma(3u^2 u_y + gu) - \gamma^2 u^2 = -u^2 \eta_{xx} (2u^2 \eta_{xx} - \gamma u + g) \quad (21)$$



there, where we have substituted the formula  $\eta_{xx} = (u_y + \gamma)/u$ , which is due to  $u_y + \gamma = v_x = (\eta_x u)_x = \eta_{xx} u$  at the crest. By Theorem 15(b),  $\eta_{xx} < 0$  at the crest. So (21) implies

$$g - \gamma u(0, \eta(0)) = -2u^2 \eta_{xx}(0, \eta(0)) > 0,$$

which is a contradiction.  $\square$

**Corollary 7.** *Under the same conditions as in Theorem 3, every nontrivial wave in  $\mathcal{C}$  satisfies  $u_x < 0$  in  $\Omega^+ \cup S^+$ , as well as*

$$\left| \frac{v}{u} \right| < \sigma \quad \text{where} \quad \sigma^2 = \max_{\bar{\Omega}} \frac{g - \gamma u}{g + \gamma u}.$$

Note that  $\sigma^2 < 1$  if  $\gamma(0) < 0$ .

*Proof.* We have already proven that  $u_x < 0$  in  $\Omega^+ \cup S^+$ . Now consider the function  $f = \alpha u^2 - v^2$  for some  $0 < \alpha \leq 1$ . Differentiating it and using (10) to eliminate  $u_y$ , we obtain

$$u f_x + v f_y = (\alpha + 1)(v^2 + u^2)u_x - \{\alpha(\gamma u + g) + \gamma u - g\}v. \quad (22)$$

We define  $\alpha = \sigma^2$ . Then, since  $\gamma u + g > 0$ , the expression in (22) is at most zero in  $\Omega^+$ . As in Lemma 4, we deduce the stated inequality on the slope.  $\square$

### 3 Existence of waves

We briefly discuss the question of existence of waves with vorticity for which there is a bound on the slope. In [CS04] the following construction of waves in  $\mathcal{W}$  is proven. Given  $c, m$ , wavelength  $2L$  and a smooth function  $\gamma \leq 0$ , there exists a connected set  $\mathcal{C}$  of waves satisfying (1), (2) and (3) such that  $\mathcal{C}$  contains exactly one trivial wave as well as a sequence of waves for which  $\sup u_n \nearrow c$ . Under the assumption  $\gamma \leq 0$ , there are no restrictions on  $c, m$ , and  $L$  for the existence of  $\mathcal{C}$  (see Example 3.4 in [Con11]). (Note however that the definition  $\gamma$  in that reference differs from ours by a sign.) Connectedness is taken in the same sense as above. (Actually in [CS04] the amount of regularity is less but the extra regularity is very easy to prove.) In fact,  $\mathcal{C}$  contains a continuous curve  $\mathcal{K}$  in function space with the same properties. This was proven later in the irrotational case in [BT03] and in the rotational case with surface tension in [Wal14].

Before discussing the relationship between Theorem 3 and  $\mathcal{C}$ , we make a few definitions related to Bernoulli's law. First, we define a function  $\Gamma \in C^3[-m, 0]$  in terms of  $\gamma$  by

$$\Gamma(-\psi) = \int_0^{-\psi} \gamma(-p) dp.$$

In terms of  $\Gamma$ , which follows from (1), Bernoulli's law can be written

$$P - P_{\text{atm}} + \frac{u^2 + v^2}{2} + g(y + d) - \Gamma(-\psi) \equiv Q \quad (23)$$

for some constant  $Q$  sometimes called the “total head”. For trivial flows with fixed vorticity function  $\gamma$  and flux  $m$ , it is shown, for instance in [CS04], that the speed squared  $\lambda = u_0^2(0)$  at the free surface and the total head  $Q$  are related by  $Q = \mathcal{Q}(\lambda)$ , where

$$\mathcal{Q}(\lambda) := \frac{\lambda}{2} + g \int_{-m}^0 \frac{ds}{\sqrt{\lambda + 2\Gamma(s)}}.$$

We easily check  $\mathcal{Q}$  is a strictly convex function of  $\lambda > -2\Gamma_{\min}$ , with a unique minimum at  $\lambda = \lambda_{\text{cr}}$ .

Now let  $\gamma$  satisfy (4), and let  $\mathcal{C}^*$  be the set of waves in  $\mathcal{C}$  such that (7) holds, which means that  $g - \gamma u > 0$  at the trough.

**Proposition 8.**

- (a)  $\mathcal{C}^*$  contains  $\mathcal{C}_{\text{loc}}$ , the part of  $\mathcal{C}$  sufficiently close to the trivial wave.
- (b) If  $\gamma(0)$  is sufficiently small, namely,

$$\gamma^2(0) < \frac{g^2}{2gL + \lambda_{\text{cr}}} \quad (24)$$

then  $\mathcal{C}^* = \mathcal{C}$ .

*Proof.* For (a) it suffices by continuity to prove that

$$g - \gamma\sqrt{\lambda} = g - \gamma u(L, \eta(L)) > 0 \quad (25)$$

holds for the trivial wave in  $\mathcal{C}$ . From [CS04] we know that  $\lambda < \lambda_{\text{cr}}$  for the trivial wave in  $\mathcal{C}$ , so it is enough to show

$$\gamma^2(0)\lambda_{\text{cr}} < g^2. \quad (26)$$

If  $\gamma(0) = 0$  then (26) is trivially satisfied, so assume that  $\gamma(0) < 0$ . Then by the convexity of  $\mathcal{Q}$  and the definition of  $\lambda_{\text{cr}}$ , (26) is equivalent to the inequality

$$0 = 2\mathcal{Q}'(\lambda_{\text{cr}}) < 2\mathcal{Q}'\left(\frac{g^2}{\gamma^2(0)}\right) = 1 - g \int_{-m}^0 \frac{dp}{(g^2/\gamma^2(0) + 2\Gamma(p))^{3/2}}. \quad (27)$$

Now by our assumptions (4) on the vorticity function  $\gamma$ ,  $\Gamma'(0) = \gamma(0) < 0$ , and moreover  $\Gamma''(p) = -\gamma'(-p) \geq 0$  so that  $\Gamma$  is convex and  $\Gamma(p) \geq \gamma(0)p$ . Thus

$$\begin{aligned} 2\mathcal{Q}'\left(\frac{g^2}{\gamma^2(0)}\right) &\geq 1 - g \int_{-m}^0 \frac{dp}{(g^2/\gamma^2(0) + 2\gamma(0)p)^{3/2}} \\ &= \frac{g}{|\gamma(0)|\sqrt{g^2/\gamma^2(0) - 2\gamma(0)m}} > 0 \end{aligned}$$

as desired.

It remains to prove (b). Since  $\mathcal{C}^* \subset \mathcal{C}$  is nonempty and  $\mathcal{C}$  is connected, it suffices to show that  $\mathcal{C}^*$  is both relatively open and relatively closed as a subset of  $\mathcal{C}$ . From its definition (see (7)),  $\mathcal{C}^*$  is clearly relatively open, so it remains to show that it is relatively closed. So consider a wave which is a limit point of  $\mathcal{C}^*$ . Since waves in  $\mathcal{C}^*$  satisfy  $|\eta_x| < 1$ , this limiting wave must have  $|\eta_x| \leq 1$ . Evaluating Bernoulli's law (23) both at the crest  $(0, \eta(0))$  and the trough  $(L, \eta(L))$ , we deduce that

$$u^2(L, \eta(L)) = u^2(0, \eta(0)) + 2g[\eta(0) - \eta(L)] \leq u^2(0, \eta(0)) + 2gL. \quad (28)$$

But it was shown in [CS04] that all the waves in  $\mathcal{C}$  satisfy  $u^2(0, \eta(0)) < \lambda_{\text{cr}}$ . So (28) implies  $u^2(L, \eta(L)) \leq \lambda_{\text{cr}} + 2gL$ . Rearranging this inequality and using the assumption (24), we obtain  $g - \gamma u(L, \eta(L)) > 0$ . That is, the limiting wave in fact lies in  $\mathcal{C}^*$ .  $\square$

Using the same argument as in the proof of Proposition 8(a), one can show that (24) holds whenever

$$\frac{1}{\sqrt{1 - 2L\gamma^2(0)/g}} - \frac{1}{\sqrt{1 - 2L\gamma^2(0)/g - 2\gamma^3(0)m/g^2}} < 1. \quad (29)$$

Moreover, (24) and (29) are equivalent when  $\gamma$  is constant.

An open problem is the following question. When  $\gamma(0)$  is large enough that (24) is violated, does (7) still hold for all waves in  $\mathcal{C}$ ? In other words, is  $\mathcal{C}^* = \mathcal{C}$  always true?

## 4 Solitary waves

In this short section we show how the arguments of Sections 2 and 3 can be modified for waves which are solitary rather than periodic. By a solitary wave we mean a solution to (1) (with  $u = U - c$ ,  $v = V$ ) in the unbounded domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 : -\infty < x < \infty, -d < y < \eta(x)\}$$

with the additional asymptotic condition that

$$D^k v \rightarrow 0, \quad \eta \rightarrow 0, \quad D^k u(x, y) \rightarrow D^k u_\infty(y) \quad \text{as } x \rightarrow \pm\infty, \quad k = 0, 1, 2, \quad (30)$$

uniformly in  $y$ , for some function  $u_\infty(y)$ . Here  $D^k$  denotes any derivative of order  $k$  with respect to  $x$  or  $y$ . As before we will assume that  $u, \eta$  are even in  $x$  and that  $v$  is odd in  $x$ . We continue to assume (2),  $\sup_\Omega u < 0$ , which in particular implies  $\max u_\infty < 0$ . We will assume the regularity  $\eta \in C_b^4(\mathbb{R})$ ,  $u, v, P \in C_b^3(\overline{\Omega})$ , and  $u_\infty \in C^3[-d, 0]$ . The notation  $C_b^k$  indicates functions whose derivatives up to order  $k$  are bounded and continuous. The topology is that of uniform convergence of those derivatives.

Letting  $\Omega^+$ ,  $S^+$ , and  $B^+$  denote the right halves of the fluid domain, free surface, and bed,

$$\begin{aligned} \Omega^+ &= \{(x, y) \in \mathbb{R}^2 : x > 0, -d < y < \eta(x)\}, \\ S^+ &= \{(x, \eta(x)) : x > 0\}, \\ B^+ &= \{(x, -d) : x > 0\}, \end{aligned}$$

(see Figure 1b), we will continue to assume that the strict monotonicity (3) holds, that is  $v > 0$  in  $\Omega^+ \cup S^+$ , for all nontrivial waves. The vorticity function  $\gamma$  is defined exactly as before, so that  $\gamma = -(u_\infty)_y$ , and will be assumed to satisfy (4).

**Definition 9** (The water wave set  $\mathcal{W}$  for solitary waves). In this section we denote by  $\mathcal{W}$  the set of waves satisfying the above assumptions, or more precisely the set of tuples  $(u, v, P, \eta, u_\infty, d)$  with  $d > 0$ ,  $\eta \in C_b^4(\mathbb{R})$ ,  $u_\infty \in C_b^3[-d, 0]$ , and  $u, v, P \in C_b^3(\overline{\Omega})$  for which  $u, P, \eta$  are even in  $x$ , and  $v$  are odd in  $x$ , which satisfy (1), (30), (2), and (4), and which either satisfy the monotonicity condition (3) or are trivial.

We also define the mapping  $\tau$  in an analogous way, namely,

$$\tau: \mathcal{W} \rightarrow (C_b^0[0, \infty))^7, \quad \tau(u, v, P, \eta, u_\infty, d)(x) = (u, v, u_x, v_x, u_y, u_{xx}, u_{xy})(x, \eta(x)).$$

**Theorem 10.** Let  $\mathcal{C} \subset \mathcal{W}$  be a set of solitary water waves such that  $\tau(\mathcal{C})$  is a connected subset of  $(C^0[0, \infty))^7$ . If  $\mathcal{C}$  contains at least one trivial wave, and if all the waves in  $\mathcal{C}$  satisfy

$$g - \gamma u_\infty(0) > 0 \quad (31)$$

then the slope  $|v/u|$  of any streamline of any wave in  $\mathcal{C}$  is  $< 1$ .

Recall that  $u_\infty(0)$  is the relative velocity of the fluid on the surface at infinity. The proof of this theorem will follow from the following lemmas.

**Lemma 11.** If a wave in  $\mathcal{W}$  satisfies  $u_x < 0$  in  $\Omega^+ \cup S^+$ , then  $\sup_{\overline{\Omega}} |v/u| < 1$ .

*Proof.* As in Lemma 4, the function  $f = \frac{1}{2}(u^2 - v^2)$  has  $f = u^2/2 > 0$  on  $\partial\Omega^+ \setminus S^+$ . By (30),  $f$  is a bounded function and  $\lim_{x \rightarrow \infty} f = u_\infty^2/2 > 0$ . Differentiating  $f$ , we find

$$uf_x + vf_y = (u^2 + v^2)u_x - \gamma uv < 0 \quad \text{on } \Omega^+ \cup S^+.$$

Suppose for the sake of contradiction that  $\min_{\overline{\Omega^+}} f \leq 0$ . Then the minimum could not be achieved on  $\partial\Omega^+ \setminus S^+$  nor at infinity, but only at some point  $(x^*, y^*)$  in  $\Omega^+ \cup S^+$ . The proof concludes exactly as in Lemma 4.  $\square$

As in Section 2 we let  $\mathcal{W}_0$  denote the set of waves in  $\mathcal{W}$  for which  $u_x \leq 0$  on  $S^+$ . The set  $\mathcal{W}_0$  contains all the trivial waves in  $\mathcal{W}$ , and  $(u, v, P, \eta, u_\infty, d)$  in  $\mathcal{W}$  lies in  $\mathcal{W}_0$  if and only if  $\tau(u, v, P, \eta, u_\infty, d)$  lies in  $\tau(\mathcal{W}_0)$ .

**Lemma 12** (Bounds along  $S^+$ ).

- (a) For every nontrivial wave in  $\mathcal{W}_0$ ,  $u_x < 0$  on  $\Omega^+ \cup S^+$ .
- (b) For every nontrivial wave in  $\mathcal{W} \setminus \mathcal{W}_0$ ,  $\inf_{\Omega^+} u_x/u < 0$  is achieved at a point  $z_0 \in S^+$  where

$$g(u^4 - 4u^2v^2 - v^4) - \gamma u^3(u^2 + 5v^2) < 0. \quad (32)$$

*Proof.* We follow the proof of Lemma 5. Consider any nontrivial wave in  $\mathcal{W}$ . As before  $w = u_x/u$  and  $s = v/u$  satisfy the elliptic equations (12) and (13), and applying the Hopf lemma to  $s$  on  $B^+$  yields  $w < 0$  there. We also have  $w = 0$  on  $x = 0$  by symmetry, and  $w \rightarrow (u_\infty)_x/u_\infty = 0$  as  $x \rightarrow +\infty$ , uniformly in  $y$ , by (30). In particular, if  $I = \inf_{\Omega^+} w < 0$ , this infimum must be achieved at some point on  $S^+$ . The rest of the proof now proceeds exactly as in the proof of Lemma 5.  $\square$

*Proof of Theorem 10.* We follow the proof of Theorem 3. As before it suffices to show that  $\tau(\mathcal{C}) \cap \tau(\mathcal{W}_0)$  is relatively open in  $(C_b^0[0, \infty))^7$ . Assume the contrary. Then as before there would exist a sequence  $(u_n, v_n, P_n, \eta_n, u_\infty, d_n) \in \mathcal{C} \setminus \mathcal{W}_0$  and  $(u, v, P, \eta, u_\infty, d) \in \mathcal{C} \cap \mathcal{W}_0$  such that

$$\tau(u_n, v_n, P_n, \eta_n, u_\infty, d_n) \longrightarrow \tau(u, v, P, \eta, u_\infty, d) \quad \text{in } (C^0[0, \infty))^7 \quad (33)$$

Note that for each  $n$ ,  $\lim_{x \rightarrow \infty} (u_n)_x/u_n = 0$  uniformly. Applying Lemma 12(b) to  $(u_n, v_n, P_n, \eta_n, d_n)$  for a fixed  $n$ , we know that the function  $u_{nx}/u_n$  achieves a negative minimum over  $\overline{\Omega_n^+}$  at a point  $z_n = (x_n, \eta_n(x_n)) \in S_n^+$ ,  $x_n > 0$ , at which we have

$$\left[ g(u_n^4 - 4u_n^2v_n^2 - v_n^4) - \gamma u_n^3(u_n^2 + 5v_n^2) \right](z_n) < 0. \quad (34)$$

Suppose first that  $x_n \rightarrow +\infty$ . Then the uniform convergence, (33) and (30), imply that

$$(u_n, v_n, u_{nx}, v_{nx}, u_{ny}, u_{nxx}, u_{nxy})(z_n) \longrightarrow (u_\infty(0), 0, 0, 0, (u_\infty)_y(0), 0, 0) \quad (35)$$

because

$$\begin{aligned} |u_n(x_n, y_n) - u_\infty(0)| &\leq |u_n(x_n, \eta_n(x_n)) - u(x_n, \eta(x_n))| + |u(x_n, \eta(x_n)) - u_\infty(0)| \\ &\leq \sup_x |u_n(x, \eta_n(x)) - u(x, \eta(x))| + |u(x_n, 0) - u_\infty(0)| + C|\eta(x_n)|, \end{aligned}$$

and similarly for the other components in (35). Taking  $n \rightarrow \infty$  in (34) therefore yields  $g^4(0) - \gamma u_\infty^5(0) \leq 0$  and hence  $g - \gamma u_\infty(0) \leq 0$ , contradicting (31).

So  $x_n$  must be bounded. By compactness we may now assume that  $x_n$  converges to some  $x_0 \geq 0$ . The uniform convergence (33) then implies that that

$$(u_n, v_n, u_{nx}, v_{nx}, u_{ny}, u_{nxx}, u_{nxy})(z_n) \longrightarrow (u, v, u_x, v_x, u_y, u_{xx}, u_{xy})(z_0) \quad (36)$$

where  $z_0 = (x_0, \eta(x_0))$  lies on the free surface. In case  $(u, v, P, \eta, u_\infty, d)$  is trivial, (36) reduces to (35) and we get a contradiction as in the preceding paragraph. So  $(u, v, P, \eta, u_\infty, d)$  must be nontrivial. As in the proof of Theorem 3, we have  $0 < u_{nx}(z_n) \rightarrow u_x(z_0) \leq 0$  as  $n \rightarrow \infty$  so that  $u_x(z_0) = 0$ . Lemma 12(a) now implies that  $z_0$  must be the crest  $(0, \eta(0))$ . Applying Theorem 17(e), we obtain

$$\frac{d}{dx} u^2(x, \eta(x)) = 2uu_x + 2vu_y = -2P_x > 0 \quad \text{for } x > 0.$$

Thus  $0 > u(0, \eta(0)) > u_\infty(0)$  and hence

$$g - \gamma u_\infty(0) \leq g - \gamma u(0, \eta(0)) = g - \gamma u(z_0) \leq 0,$$

contradicting (31).  $\square$

**Corollary 13.** *Under the same conditions as in Theorem 10, every nontrivial wave in  $\mathcal{C}$  satisfies  $u_x < 0$  in  $\Omega^+ \cup S^+$ , as well as*

$$\left| \frac{v}{u} \right| < \sigma \quad \text{where } \sigma^2 = \max_{\bar{\Omega}} \frac{g - \gamma u}{g + \gamma u}.$$

*Proof.* The proof is exactly the same as the proof of Corollary 7 for periodic waves, with Lemma 4 replaced by Lemma 11.  $\square$

In [Whe13, Whe15], the following construction of solitary waves in  $\mathcal{W}$  is proven. Given  $c, d$ , and a smooth negative function  $u_\infty^*$  of  $-d \leq y \leq 0$  with  $(u_\infty^*)_y \geq 0$  and satisfying the normalization condition

$$g \int_{-d}^0 \frac{dy}{(u_\infty^*)^2(y)} = 1,$$

there exists a connected set  $\mathcal{C}$  of waves satisfying (1), (2), (3), and (30) such that  $\mathcal{C}$  contains exactly one trivial wave as well as a sequence of waves for which  $\sup u_n \nearrow c$ . Here the asymptotic horizontal velocity  $u_\infty(y)$  in (30) is given by

$$u_\infty(y) = F u_\infty^*(y)$$

for some positive non-dimensional parameter  $F$ , called the Froude number, which varies along  $\mathcal{C}$ . The trivial wave in  $\mathcal{C}$  has  $F = 1$ , while the nontrivial waves in  $\mathcal{C}$  have  $1 < F < 2$ .

Now suppose that  $u_0 = u_\infty^*$  satisfies (5), in which case the vorticity function  $\gamma$  of any wave in  $\mathcal{C}$  satisfies (4). Let  $\mathcal{C}^*$  be the set of waves in  $\mathcal{C}$  such that (31) holds, which means that  $g - \gamma u_\infty(0) > 0$ .

**Proposition 14.**

- (a)  $\mathcal{C}^*$  contains  $\mathcal{C}_{\text{loc}}$ , the part of  $\mathcal{C}$  sufficiently close to the trivial wave.
- (b)  $\mathcal{C}^*$  contains all waves in  $\mathcal{C}$  satisfying the bound

$$F^2 < \frac{g}{|(u_\infty^*)_y u_\infty^*(0)|}. \quad (37)$$

- (c) If  $u_\infty^*$  satisfies

$$|(u_\infty^*)_y u_\infty^*(0)| < \frac{g}{4} \quad (38)$$

then (37) always holds, so that  $\mathcal{C}^* = \mathcal{C}$ .

*Proof.* For (a) it suffices by continuity to prove that (31) holds for the trivial wave in  $\mathcal{C}$ . From [Whe13] we know that this wave has, in the notation of Section 3,  $\lambda = \lambda_{\text{cr}}$ , so the rest of the proof proceeds exactly as in the proof of Proposition 8(a). To prove (b), we simply notice that, by the scaling  $u_\infty = F u_\infty^*$  and the definition of  $\gamma$ , we have

$$g - \gamma u_\infty(0) = g + [(u_\infty)_y u_\infty](0) = g + F^2 [(u_\infty^*)_y u_\infty^*](0) = g - F^2 |(u_\infty^*)_y(0) u_\infty^*(0)|.$$

The remaining statement (c) then follows immediately from (b) and the inequality  $F < 2$  in [Whe15].  $\square$

In the case of constant vorticity, the dimensionless vorticity  $\tilde{\gamma} := \gamma d / |u_\infty(0)|^{1/2} < 0$  is constant along  $\mathcal{C}$ , and (38) is equivalent to  $|\tilde{\gamma}| < 1/3$ .

## 5 Some general inequalities on the pressure

### 5.1 Periodic case

Some of the following facts are already known under various assumptions but others appear to be new. In fact, under certain assumptions, versions of (a) and (f) appear in [Var09], and a version of (c) appears in [Con14].

In this section, for the sake of generality, we assume only that the free surface is a  $C^2$  curve which does not self-intersect, along which  $y > -d$ , and which is horizontally periodic with period  $2L$ . We let  $\Omega$  be the region between  $S$  and  $B = \{y = -d\}$ , and require that the equations (1) hold in  $\Omega$ , with the kinematic boundary condition (1e) replaced by the condition that  $(u, v)$  is tangent to  $S$ . Here we continue to denote  $u = U - c$ ,  $v = V$  and we define the vorticity to be  $\omega = v_x - u_y$ . We assume the regularity and periodicity  $u, v \in C_{\text{per}}^1(\overline{\Omega})$  and  $P \in C_{\text{per}}^2(\overline{\Omega})$ , where as before “per” denotes  $2L$ -periodicity in  $x$ .

We do *not* assume that  $S$  is a graph. Thus *the waves could be overturning*. Nor do we assume that the wave is symmetric in any way, or that (2) or the monotonicity conditions (3) hold. Finally, we *not* assume that  $\omega$  is a function of  $\psi$ , nor that it has any particular sign.

Define

$$\eta_{\max} = \max_S y, \quad \eta_{\min} = \min_S y.$$

Any point on  $S$  for which  $y = \eta_{\max}$  is called a *crest*, while any point on  $S$  for which  $y = \eta_{\min}$  is called a *trough*.

As before, incompressibility allows us to define a stream function  $\psi$ , which by the boundary conditions can be taken to vanish on the free surface  $S$  and to be equal to some other constant  $m$ , not necessarily positive, on the bed  $B$ .

**Theorem 15.** Consider any nontrivial solution (that is, with  $S$  not a horizontal line) to (1) in  $\Omega$  in the above sense with  $u^2 + v^2 \neq 0$ .

(a) The pressure satisfies

$$g\eta_{\min} \leq P - P_{\text{atm}} + gy \leq g\eta_{\max}, \quad (39)$$

with equality only at crests or troughs.

(b) The free surface is strictly concave at any crest and strictly convex at any trough.

(c)  $\eta_{\max} - \eta_{\min} > \frac{1}{g}(\max_B P - \min_B P)$ .

(d)  $P > P_{\text{atm}}$  at all depths below the troughs.

(e) If  $\omega u + g \geq 0$ , then  $P \geq P_{\text{atm}}$  with equality only on the free surface  $S$ , on which  $\partial P / \partial n < 0$ .

(f) If  $\omega_{\max}(u^2 + v^2) - 4gu \geq 0$ , then  $P + \frac{1}{2}\omega_{\max}\psi \geq P_{\text{atm}}$  with equality only on the free surface. For instance, this is true if  $\omega_{\max} > 0$  and  $u \leq 0$ .

(g) If  $\omega \geq 0$  in the fluid, while  $u < 0$  at either a crest or a trough, then the relative speed  $\sqrt{u^2 + v^2}$  in the fluid is maximized at all troughs.

(h) If  $\omega \leq 0$  in the fluid, while  $u < 0$  at either a crest or a trough, then the relative speed  $\sqrt{u^2 + v^2}$  in the fluid is minimized at all crests.

*Proof.* (a) Consider the function  $f = P + gy$ . Since  $f = P_{\text{atm}} + gy$  on the free surface, clearly it is nonconstant. A straightforward though tedious calculation shows that

$$(u^2 + v^2)\Delta f + 2(f_x - \omega v)f_x + 2(f_y + \omega u)f_y = 0. \quad (40)$$

Since  $u^2 + v^2 \neq 0$ , the maximum principle implies that  $f$  can only achieve its maximum and minimum values on the boundary. Since on the bed  $y = -d$  we have  $v = 0$  and

$$f_y = P_y + g = -uv_x - vv_y = 0,$$

the Hopf lemma implies that the maximum and minimum of  $f$  must be achieved on the free surface. But the pressure  $P$  is constant on the free surface, so  $f$  is maximized at all crests and minimized at all troughs. Rewriting this in terms of the pressure, we obtain (39) as desired.

(d) From the lower bound in (39), we see that

$$P - P_{\text{atm}} \geq g(\eta_{\min} - y),$$

and hence in particular that  $P > P_{\text{atm}}$  at all depths below the troughs, which is (d).

(c) Evaluating the inequalities in (39) at the points along the bed where the pressure  $P$  is maximized and minimized, we also find

$$\max_B P - P_{\text{atm}} - gd < g\eta_{\max}, \quad \min_B P - P_{\text{atm}} - gd > g\eta_{\min}.$$

Subtracting these two inequalities yields (c).

(b) Consider any crest. In a neighborhood of the crest,  $y$  is locally solvable as a function of  $x$ . [Indeed, let  $S$  be parametrized as  $(\alpha(s), \beta(s))$  where  $(\alpha')^2 + (\beta')^2 \neq 0$ . At the crest  $\beta' = 0$  so that  $\alpha' \neq 0$ . So we can locally solve  $y = \beta(\alpha^{-1}(x)) =: \eta(x)$ .] Now at the crest  $\eta_x = 0$  so we must have

$v = 0$  and  $u \neq 0$  due to our assumption  $u^2 + v^2 \neq 0$ . Applying the Hopf lemma to  $f$  at the crest, we find that

$$0 < f_y = P_y + g = -uv_x = -u^2\eta_{xx}$$

and hence that  $\eta_{xx} < 0$ . Thus the free surface is strictly concave at that point. An analogous argument shows that  $\eta_{xx} > 0$  at all troughs. The emphasis here is on the strictness of the inequalities.

(e) We note from (40) and the assumption  $\omega u + g \geq 0$  that

$$(u^2 + v^2)\Delta P + 2(P_x - \omega v)P_x + 2(P_y + \omega u + 2g)P_y = -2g(\omega u + g) \leq 0$$

in  $\Omega$  and hence that  $P$  achieves its minimum either on the free surface or on the bed  $y = -d$ . On  $y = -d$  we have  $P_y = -g < 0$ , so the minimum cannot occur there. Thus  $P$  is minimized on the free surface, where it is constant. By the Hopf lemma,  $\partial P / \partial n < 0$  along the free surface  $S$  where the normal  $n$  points away from  $\Omega$ .

(f) Consider the function  $f = P + \frac{1}{2}\omega_{\max}\psi$ . Yet another direct and somewhat tedious calculation shows that  $f$  satisfies

$$2(u^2 + v^2)\Delta f + 4af_x + 4bf_y = -(\omega_{\max} - \omega)(\omega_{\max}(u^2 + v^2) - 4gu) \leq 0,$$

where the coefficients  $a$  and  $b$  are

$$a = f_x + (\omega_{\max} - \omega)v, \quad b = f_y - (\omega_{\max} - \omega)u + 2g.$$

So by the maximum principle  $f$  can only be minimized on the free surface or the bed  $y = -d$ . On the bed,

$$f_y = P_y + \frac{1}{2}\omega_{\max}v = -g < 0,$$

so the minimum cannot occur there. Thus  $f$  is minimized on the free surface where it takes the constant value  $P_{\text{atm}}$ .

(g) and (h). By our assumption  $u^2 + v^2 \neq 0$ , the stream function  $\psi$  has no critical points in  $\Omega$ , so it must attain its maximum or its minimum on the free boundary, where it is constant. By assumption there is a crest or trough where  $u < 0$ , so that  $\psi_x = v = 0$  there while  $\psi_y = \partial\psi/\partial n = u < 0$ . Thus  $\psi$  must be minimized along the free surface where it vanishes, and maximized along the bed where its value is  $m > 0$ . Now consider the function  $\Gamma(x, y)$  defined by

$$\Gamma(x, y) = P - P_{\text{atm}} + \frac{u^2 + v^2}{2} + gy - K.$$

This is a generalization of the function  $\Gamma$  of Section 3. Note that  $\Gamma_x = \omega v$ ,  $\Gamma_y = -\omega u$ , and hence  $u\Gamma_x + v\Gamma_y = 0$ . Thus  $\Gamma$  is constant along streamlines. In particular,  $\Gamma$  is constant along the free surface and also along the bed, and we can choose the constant  $K$  so that  $\Gamma$  vanishes on the free surface.

We claim that if  $\omega \geq 0$  in  $\Omega$ , then  $\Gamma \leq 0$  in  $\Omega$ , and similarly if  $\omega \leq 0$  then  $\Gamma \geq 0$ . To prove this, choose any  $z_0 \in \Omega$ . Let  $\psi_0 = \psi(z_0)$  and solve the ODE

$$\dot{z}(s) = \frac{\nabla\psi}{|\nabla\psi|^2}(z(s)), \quad z(0) = z_0,$$



where the denominator  $|\nabla\psi|^2 = u^2 + v^2 \neq 0$ . We easily check that  $\frac{d}{ds}\psi(z(s)) = 1$ , so in particular  $\psi(z(-\psi_0)) = 0$  and  $\psi(z(m - \psi_0)) = m$ . That is,  $z(-\psi_0)$  lies on the free surface while  $z(m - \psi_0)$  lies on the bed. Thus we must have  $\Gamma(z(-\psi_0)) = 0$ . Hence

$$\begin{aligned}\Gamma(z_0) &= \Gamma(z(0)) = \Gamma(z(-\psi_0)) + \int_{-\psi_0}^0 \frac{d}{ds}\Gamma(z(s)) ds = \int_{-\psi_0}^0 \nabla\Gamma \cdot \frac{\nabla\psi}{|\nabla\psi|^2}(z(s)) ds \\ &= \int_{-\psi_0}^0 \frac{\omega v\psi_x - \omega u\psi_y}{|\nabla\psi|^2}(z(s)) ds = - \int_{-\psi_0}^0 \omega(z(s)) ds.\end{aligned}$$

Thus  $\Gamma(z_0) \geq 0$  if  $\omega \leq 0$ , while  $\Gamma(z_0) \leq 0$  if  $\omega \geq 0$ .

Now let  $z^* = (x^*, \eta(x^*))$  be a trough, so that  $\Gamma(z^*) = 0$ . Suppose  $\omega \geq 0$  in  $\Omega$ , so that  $\Gamma \leq 0$ . Then at any point  $(x, y) \in \overline{\Omega}$  we have from Bernoulli's law that

$$\begin{aligned}(u^2 + v^2)(x, y) &\leq (u^2 + v^2 - 2\Gamma(-\psi))(x, y) = C - 2P - 2gy \\ &\leq C - 2P_{\text{atm}} - 2g\eta(x^*) \leq (u^2 + v^2 - 2\Gamma(-\psi))(z^*) = (u^2 + v^2)(z^*)\end{aligned}$$

since  $\Gamma(-\psi(0)) = \Gamma(0) = 0$ . Similarly, if  $z^*$  is a crest and  $\omega \leq 0$  in  $\Omega$ , then  $\Gamma \geq 0$ , so that at any point  $(x, y) \in \overline{\Omega}$  we have

$$(u^2 + v^2)(x, y) \geq (u^2 + v^2 - 2\Gamma(-\psi))(x, y) \geq (u^2 + v^2 - 2\Gamma(-\psi))(z^*) = (u^2 + v^2)(z^*). \quad \square$$

**Proposition 16** (Overturning periodic waves must have a pressure sink). *Consider a periodic wave as above and suppose that  $u^2 + v^2 \neq 0$  on the free surface. If the wave overturns, meaning that  $u$  takes both positive and negative values along the free surface  $S$ , then there is a point on the free surface where  $\partial P/\partial n > 0$ . In particular, the pressure  $P$  achieves its minimum value inside the fluid and not on the free surface. Thus by Theorem 15(e) there is either a stagnation point in the fluid or a point where  $\omega u + g < 0$ .*

*Proof.* Since  $u^2 + v^2 \neq 0$  along  $S$ , the vector  $(v, -u)$  is normal to  $S$  and either points into the fluid or out of the fluid. Without loss of generality we assume below that it points out of the fluid, which implies that  $u < 0$  at crests and troughs. Also since  $u^2 + v^2 \neq 0$  along  $S$ , we can think of the free surface as being parametrized by a curve  $(x, y) = (X(s), Y(s))$ , where  $X$  and  $Y$  are solutions of the ordinary differential equation  $X'(s) = u(X(s), Y(s))$ ,  $Y'(s) = v(X(s), Y(s))$ . Since the wave is periodic,  $X'$  and  $Y'$  are periodic functions of  $s$  with some period  $T$ . Defining the angle  $\theta(s)$  that the free surface makes with the horizontal in the usual way, we have

$$\cos \theta = \frac{X'}{\sqrt{(X')^2 + (Y')^2}}, \quad \sin \theta = \frac{Y'}{\sqrt{(X')^2 + (Y')^2}}.$$

Because of the periodicity of  $X'$  and  $Y'$ , we know that  $\theta(s + T) = \theta(s) + 2\pi m$  for some integer  $m$ , which must be constant by continuity. In fact, since the free surface does not self-intersect, this constant  $m$  must vanish, so that  $\theta$  is periodic. This can be seen by considering the tangent angle of the simple closed curve formed by drawing a horizontal line between two consecutive crests. Differentiating along the free surface, we easily check that

$$X'' = uu_x + vv_y = -P_x, \quad Y'' = uv_x + vv_y = -P_y - g, \quad \theta' = \frac{Y'X'' - X'Y''}{(X')^2 + (Y')^2}. \quad (41)$$

Now consider a wave which overturns and assume for the sake of contradiction that  $\partial P/\partial n \leq 0$  along the whole free surface. Since the normal vector  $(v, -u)$  on  $S$  points out of the fluid, we must have

$$\begin{aligned} 0 &\geq vP_x - uP_y = -Y'X'' + X'Y'' + gX' \\ &= -((X')^2 + (Y')^2)\theta' + g\sqrt{(X')^2 + (Y')^2} \cos \theta \end{aligned}$$

all along the free surface by (41). Rearranging, we find

$$\theta' \geq g \frac{\cos \theta}{\sqrt{(X')^2 + (Y')^2}}. \quad (42)$$

In particular, along  $S$  we have

$$\theta'(s) > 0 \text{ whenever } \cos \theta(s) > 0. \quad (43)$$

Because the wave overturns, there exists  $s_0$  for which  $u(X(s_0), Y(s_0)) > 0$  and hence  $\cos \theta(s_0) > 0$ . Define

$$s_1 = \inf\{s > s_0 : \theta(s) = \theta(s_0)\}.$$

By periodicity,  $s_1 \leq s_0 + T$ . Since  $\cos \theta(s_1) = \cos \theta(s_0) > 0$ , (43) implies

$$\theta'(s_0) > 0, \quad \theta'(s_1) > 0,$$

so that  $s_1 > s_0$ . Thus  $\theta(s)$  is strictly increasing near both  $s_0$  and  $s_1$ , so there must be another root of the equation  $\theta(s) = \theta(s_0)$  between  $s_0$  and  $s_1$ . This contradicts the definition of  $s_1$ .  $\square$

## 5.2 Solitary case

As above, we assume only that the free surface is a  $C^2$  curve which does not self-intersect and along which  $y > -d$ , and let  $\Omega$  be the region between  $S$  and  $B = \{y = -d\}$ . We require that the equations (1) hold in  $\Omega$ , with the kinematic boundary condition (1e) replaced by the condition that  $(u, v)$  is tangent to  $S$ . Instead of horizontal periodicity, we require that  $S$  is a graph  $y = \eta(x)$  for  $|x|$  sufficiently large and that

$$\eta \rightarrow 0, \quad P - P_{\text{atm}} + gy \rightarrow 0 \quad \text{as } x \rightarrow \pm\infty, \quad (44)$$

uniformly in  $y$ . The asymptotic condition (44) is implied by (30). As for regularity, we assume that  $u, v, P \in C_b^1(\overline{\Omega})$  while  $P \in C_b^2(\overline{\Omega})$ .

We define

$$\eta_{\max} = \sup_S y, \quad \eta_{\min} = \inf_S y.$$

Any point on  $S$  for which  $y = \eta_{\max}$  is called a *crest*, while any point on  $S$  for which  $y = \eta_{\min}$  is called a *trough*. If  $\eta_{\max}$  is achieved at  $(\pm\infty, 0)$ , we also call  $(\pm\infty, 0)$  a “crest”. If  $\eta_{\min}$  is achieved at  $(\pm\infty, 0)$ , we also call  $(\pm\infty, 0)$  a “trough”.

As before, incompressibility allows us to define a stream function  $\psi$ , which by the boundary conditions can be taken to vanish on the free surface  $S$  and to be equal to some other constant  $m$ , not necessarily positive, on the bed  $B$ .

**Theorem 17.** Consider any nontrivial solution to (1) in  $\Omega$  as above, with  $u^2 + v^2 \geq \delta > 0$  for some constant  $\delta$ . Then

(a) The pressure satisfies

$$g\eta_{\min} \leq P - P_{\text{atm}} + gy \leq g\eta_{\max}, \quad (45)$$

with equality only at finite crests or troughs (and in the limit as  $x \rightarrow \pm\infty$ ).

(b) The free surface is strictly concave at any finite crest and strictly convex at any finite trough.

(c)  $\eta_{\max} - \eta_{\min} > \frac{1}{g}(\sup_B P - \inf_B P)$ .

(d)  $P > P_{\text{atm}}$  at all depths below the troughs.

(e) If  $\omega u + g \geq 0$ , then  $P \geq P_{\text{atm}}$  with equality only on the free surface  $S$ , on which  $\partial P / \partial n < 0$ .

(f) Suppose that  $\omega_{\max}(u^2 + v^2) - 4gu \geq 0$  and, in addition to (44), we have

$$P_y \rightarrow -g, \quad v \rightarrow 0 \quad \text{as } x \rightarrow \pm\infty, \quad (46)$$

uniformly in  $y$ . Then  $P + \frac{1}{2}\omega_{\max}\psi \geq P_{\text{atm}}$  with equality only on the free surface. For instance, this is true if  $\omega_{\max} > 0$ ,  $u \leq 0$ , and (30) holds.

(g) If  $\omega \geq 0$  in the fluid while  $u < 0$  for  $|x|$  sufficiently large, then the relative speed  $\sqrt{u^2 + v^2}$  in the fluid is maximized at all troughs.

(h) If  $\omega \leq 0$  in the fluid while  $u < 0$  for  $|x|$  sufficiently large, then the relative speed  $\sqrt{u^2 + v^2}$  in the fluid is minimized at all crests.

*Proof.* The proof is almost identical to the periodic case so we only indicate the differences.

(a) By assumption,  $f = P + gy \rightarrow P_{\text{atm}}$  as  $x \rightarrow \pm\infty$ , uniformly in  $y$ . The Hopf lemma implies that the supremum and infimum of  $f$  can only be achieved on the free surface or in the limit as  $x \rightarrow \pm\infty$ . If both are achieved on the free surface then the argument proceeds exactly as before, so suppose that the infimum of  $f$  is achieved as  $x \rightarrow \pm\infty$  but not at any finite point on the free surface. Then  $f > P_{\text{atm}}$  at all points in the fluid domain and along the free surface. Restricting  $f$  to the free surface we see that  $\eta_{\min} = 0$  is also achieved as  $x \rightarrow \pm\infty$ . In particular,  $(\pm\infty, 0)$  is a trough by our above definition and the inequality  $f > P_{\text{atm}}$  can be rewritten as  $P - P_{\text{atm}} + gy > g\eta_{\min}$ . Similarly, if the supremum of  $f$  is achieved as  $x \rightarrow \pm\infty$  but not at any point along the free surface then we must have  $P - P_{\text{atm}} + gy < g\eta_{\min}$ .

(d) No change.

(c) Replace max and min by sup and inf. The inequality is still strict since by (44) the sup and inf of the restriction of  $P$  to the bed  $B$  cannot both be achieved as  $x \rightarrow \pm\infty$ .

(b) Consider any finite crest or trough.

(e) Replacing min by inf, suppose that the infimum of  $P$  is achieved as  $x \rightarrow \pm\infty$ . Then  $P + gy \rightarrow P_{\text{atm}}$  as  $x \rightarrow \pm\infty$  implies  $\inf P = P_{\text{atm}}$  as before.

(f) Thanks to (46), we have

$$f_y = P_y + \frac{1}{2}\omega_{\max}v \rightarrow -g < 0$$

as  $x \rightarrow \pm\infty$ , uniformly in  $y$ . Thus if the infimum of  $f$  is achieved as  $x \rightarrow \pm\infty$ , it must be achieved at  $(\pm\infty, 0)$  where  $f$  takes the value  $P_{\text{atm}}$ .

(g) and (h). Since  $\psi$  has no critical points and  $\psi_y = u < 0$  for  $|x|$  sufficiently large, we find as before that  $\psi$  is minimized along the free surface where it vanishes and maximized along the bed where its value is  $m > 0$ . If  $z^* = (\pm\infty, 0)$ , we simply take limits.  $\square$

**Proposition 18** (Overturning solitary waves must have a pressure sink). *Consider a solitary wave as above and suppose that  $u^2 + v^2 \geq \delta > 0$  on the free surface for some constant  $\delta$ , and  $\eta_x \rightarrow 0$  as  $x \rightarrow \pm\infty$ . If the wave overturns, meaning that  $u$  takes both positive and negative values along the free surface  $S$ , then there is a point on the free surface where  $\partial P/\partial n > 0$ . In particular, the pressure  $P$  achieves its minimum value inside the fluid and not on the free surface.*

*Proof.* Again, the proof is almost identical to the periodic case so we only indicate the differences. Assuming as before that  $(v, -u)$  points out of the fluid, we have  $e^{i\theta(s)} \rightarrow -1$  as  $s \rightarrow \pm\infty$ . Without loss of generality, we may assume that  $\theta(s) \rightarrow \pi$  as  $s \rightarrow +\infty$ . To see that  $\theta(s) \rightarrow \pi$  as  $s \rightarrow -\infty$  as well, we consider the tangent angle of the oriented boundary of  $\Omega \cap \{|x| < M\}$  for  $M$  sufficiently large.

Now consider a wave which overturns and assume for the sake of contradiction that  $\partial P/\partial n \leq 0$  along the whole free surface. As before we find

$$\theta'(s) > 0 \text{ whenever } \cos \theta(s) > 0. \quad (47)$$

Since  $\theta \rightarrow \pi$  as  $s \rightarrow \pm\infty$ , there must exist values  $s_0 < s_2$  for which  $\theta(s_0) = \theta(s_2)$  and  $u(X(s_0), Y(s_0)) > 0$  and hence  $\cos \theta(s_0) = \cos \theta(s_2) > 0$ . Let

$$s_1 = \inf\{s > s_0 : \theta(s) = \theta(s_0)\}.$$

By construction,  $s_1 < s_2$ , so in particular  $s_1$  is finite, and we reach a contradiction exactly as in the periodic case.  $\square$

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